

# Optimal trading with a trailing stop

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## Abstract

Trailing stop is a popular stop-loss trading strategy by which the investor will sell the asset once its price experiences a pre-specified percentage drawdown. In this paper, we study the problem of timing buy and then sell an asset subject to a trailing stop. Under a general linear diffusion framework, we study an optimal double stopping problem with a random path-dependent maturity. Specifically, we first derive the optimal liquidation strategy prior to a given trailing stop, and prove the optimality of using a sell limit order in conjunction with the trailing stop. Our analytic results for the liquidation problem is then used to solve for the optimal strategy to acquire the asset and simultaneously initiate the trailing stop. The method of solution also lends itself to an efficient numerical method for computing the optimal acquisition and liquidation regions. For illustration, we implement an example and conduct a sensitivity analysis under the exponential Ornstein-Uhlenbeck model.

**Keywords:** trailing stop, stop loss, optimal stopping, drawdown, stochastic floor

**JEL Classification:** C41, C61, G11

**Mathematics Subject Classification (2010):** 60G40, 62L15, 91G20, 91G80

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# 1 Introduction

Trailing stops are a popular trade order widely used by proprietary traders and retail investors to provide downside protection for an existing position. In contrast to a fixed stop-loss exit, a trailing stop is characterized by a stochastic floor that moves in parallel to the running maximum of the asset price. A trailing stop is triggered when the prevailing price of an asset falls below the stochastic floor. In essence, it allows an investor to specify a limit on the maximum possible loss while not limiting the maximum possible gain. The downside protection is also dynamic as the stochastic floor is raised whenever the asset price moves upward.

In addition to setting a trailing stop order, the investor can also use a limit order to sell at certain price target. Indeed, if the price is sufficiently high, the investor may prefer to sell immediately as opposed to waiting to set off the trailing stop. The investor's position will be liquidated by either order. In this paper, we investigate the optimal timing to liquidate a position subject to a trailing stop. Mathematically, we recognize the trailing stop as a timing constraint in the sense that it installs a path-dependent random maturity into the liquidation problem, rendering the problem significantly more difficult to solve. Furthermore, the investor can also decide when to establish the position. This leads us to analyze the optimal timing to enter the market. In sum, we study an optimal double stopping problem subject to a trailing stop. By using excursion theory of linear diffusion, we derive the value functions using the smallest concave majorant characterization, and discuss the effect of trailing stopping on the optimal trading strategies analytically and numerically. Among our results, we reduce the problem of finding the optimal timing strategies to solving an ODE problem, which forms the basis of our numerical scheme in determining the optimal asset acquisition and liquidation regions.

In general, a trailing stop can be defined as the first time when the asset price  $X$  drops below  $f(\overline{X})$ , where  $\overline{X}$  is the running maximum process of  $X$ , and  $f$  is an increasing function such that  $f(x) < x$  for all  $x$  in the support of  $X$ . In applied probability literature, such a stopping time is related to the drawdown process and its first passage time. We refer to Lehoczky (1977), Zhang (2015), and Zhang and Hadjiladis (2012), for a partial list of studies on drawdowns under linear diffusions. Moreover, the optimality of trailing stops in exercising Russian options and detecting abrupt changes can be found in Shepp and Shiryaev (1993) and Zhang et al. (2015), respectively.

Despite being commonly used by practitioners, trailing stops have been scarcely studied in the mathematical finance literature. We trace back to Glynn and Iglehart (1995), who studied the expected discounted reward at a trailing stop under a discrete-time random walk or a geometric Brownian motion (GBM) model, and found that it would be optimal to never use the trailing stop if the stock followed a GBM with a positive drift. In contrast, our study is conducted in a more general linear diffusion framework, and provides concrete illustrative example on how the use of a trailing stop will affect the optimal timing to sell an asset under the exponential Ornstein-Uhlenbeck model. In a random walk model, Warburton and Zhang (2006) performed a probabilistic analysis of a variant of trailing stop. Yin et al. (2010) implemented a stochastic approximation scheme to determine the optimal percentage trailing stop level that maximizes the expected discounted simple return from liquidation. The recent study by Imkeller and Rogers (2014) compared the performance of a number of trading rules with fixed and trailing stops under an arithmetic Brownian motion model. Compared to these works, we tackle the trading problem from an optimal stopping perspective, and rigorously derive the optimal trading strategy. Mathematically, we introduce a new optimal double stopping problem subject to a stopping time constraint induced by the trailing stop. Our method of solution applies to a general linear diffusion framework, and our analytical results are amenable to computation of the value function and optimal timing strategies (see Section 5).

The incorporation of a trailing stop can be viewed as introducing a random maturity or stopping time constraint to the optimal stopping problem, in the sense that any admissible stopping time must come before triggering the trailing stop. Related studies by the authors include optimal stopping problems with maturities determined by an occupation time (Rodosthenous and Zhang (2016, 2017)) or by a default time (Leung and Yamazaki (2013)), and optimal mean reversion trading with a fixed stop-loss exit (Leung and Li (2015)). In particular, part of our study (Section 3) generalizes the analytical framework of Leung and Li (2015) to general linear diffusions, and the results from optimal stopping subject to a fixed stop-loss exit will prove to be directly useful for solving the analogous problem with a trailing stop.

The remaining of the paper is structured as follows. Section 2 presents stochastic framework for our trading problem. In Section 3, we study an optimal trading problem with a fixed stop-loss. Then, in Section 4, we study the optimal stopping problems for trading with a trailing stop. To illustrate our analytical results, we consider trading under the exponential Ornstein-Uhlenbeck model, and numerically compute the optimal acquisition and liquidation regions in Section 5. We also provide a sensitivity analysis on the optimal trading strategies with respect to model parameters.

## 2 Model Formulation

Let us consider a risky asset value process  $X = \{X_t\}_{t \geq 0}$  modeled by a linear diffusion on  $I \equiv (l, r) \subset \mathbb{R}$  with the infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x}, \quad \forall x \in I, \quad (1)$$

where  $(\mu(\cdot), \sigma(\cdot))$  is a pair of real-valued functions on  $I$  such that

$$\frac{1 + |\mu(\cdot)|}{\sigma^2(\cdot)} \in \mathbb{L}_{\text{Loc}}^1(I) \quad \text{and} \quad \sigma(x) > 0, \quad \forall x \in I.$$

For any  $\bar{x} \in I$ , the running maximum of  $X$  is denoted by

$$\overline{X}_t := \bar{x} \vee \sup_{s \in [0, t]} X_s, \quad t \geq 0.$$

We denote the unique probability law of  $X$  by  $\mathbb{P}_{x, \bar{x}}$  given  $\{X_0 = x, \overline{X}_0 = \bar{x}\}$  for any  $x, \bar{x} \in I$  with  $x \leq \bar{x}$ . The expectation associated with  $\mathbb{P}_{x, \bar{x}}$  is denoted by  $\mathbb{E}_{x, \bar{x}}$ . In calculations and results where the initial value  $\overline{X}_0 = \bar{x}$  is irrelevant, we simply write  $\mathbb{P}_x$  and  $\mathbb{E}_x$  to denote the probability law of  $X$  and the associated expectation given  $\{X_0 = x\}$ . Throughout, we assume that the upper boundary  $r$  is natural and the lower boundary  $l$  is either natural or absorbing.

We consider an investor who holds long one unit of the risky asset  $X$ . Our objective is to investigate the optimal trading strategy with a trailing stop. To this end, we consider the problem of optimal early liquidation of this risky asset, given a pre-specified trailing stop mandatory liquidation order. Specifically, we will model liquidation time by a stopping time  $\tau$  of the underlying process  $X$ , and the reward to be realized upon liquidation by  $h(X_\tau)$ , where  $h(\cdot)$  is a real-valued function on  $I$ , such that  $\{x \in I : h(x) > 0\} \neq \emptyset$ . Fix a function  $f(\cdot)$  on  $I$ , such that

$$\begin{aligned} f(\cdot) \text{ is continuous, strictly increasing on } I, \\ \text{for all } x \in I, f(x) \in I, f(x) < x. \end{aligned} \quad (2)$$

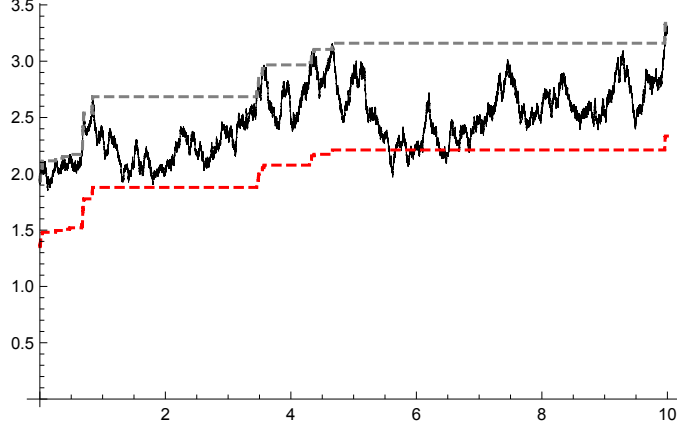


Figure 1: Sample paths of the asset price (solid black), its running maximum (gray dashed), and the 30%-drawdown floor representing the trailing stop (red dashed).

Then, we define the *stochastic floor* by  $f(\bar{X})$ , where  $\bar{X}$  is the running maximum of  $X$ . The trailing stop, denoted by  $\rho_f$ , is defined as the first time the asset value  $X$  reaches the stochastic floor  $f(\bar{X})$  from above. That is,<sup>1</sup>

$$\rho_f := \inf\{t > 0 : X_t < f(\bar{X}_t)\}. \quad (3)$$

Therefore, the investor faces the following optimal stopping problem:

$$v_f(x, \bar{x}) := \sup_{\tau \in \mathcal{T}_f^\top} \mathbb{E}_{x, \bar{x}}(e^{-q\tau} h(X_\tau) \mathbf{1}_{\{\tau < \infty\}}). \quad (4)$$

where  $q > 0$  is a subjective discounting rate, and  $\mathcal{T}_f^\top$  is the set of all stopping times of  $X$  that stop no later than the trailing stop  $\rho_f$ . Notice that  $\rho_f$  puts a mandatory selling order of the risky asset, pre-specified by the investor.

To quantify the gain in terms of expected discounted reward from liquidating earlier than the trailing stop time  $\rho_f$ , we define the *early liquidation premium* by the difference

$$p_f(x, \bar{x}) := v_f(x, \bar{x}) - g_f(x, \bar{x}), \quad (5)$$

where the second term represents the expected discounted reward from waiting to sell at the trailing stop, that is,

$$g_f(x, \bar{x}) := \mathbb{E}_{x, \bar{x}}(e^{-q\rho_f} h(X_{\rho_f}) \mathbf{1}_{\{\rho_f < \infty\}}). \quad (6)$$

As a convention, we define  $\infty - \infty = \infty$  if both terms on the right-hand side of (5) are infinity. Clearly, we have  $p_f(x, \bar{x}) \geq 0$  for all  $x, \bar{x} \in I$  with  $x \leq \bar{x}$ . For our study, the early liquidation premium turns out to be amenable to analysis and give intuitive interpretations. The related concepts of early/delayed exercise/purchase premium have been analyzed in pricing American options (see Carr et al. (1992)) and derivatives trading (Leung and Ludkovski (2011)), among other applications.

**Remark 2.1.** We give two standard choices of the floor function  $f(\cdot)$  here with  $I = \mathbb{R}$ . For example, setting  $f(x) = x - a$  for some  $a > 0$  gives the absolute drawdown floor, and  $\rho_f$  is the first time  $X$  falls from its maximum  $\bar{X}$  by  $a$  units. Another specification,  $f(x) = (1 - \alpha)x$  for some  $\alpha \in (0, 1)$ , gives the percentage

<sup>1</sup>As usual, we set  $\inf \emptyset = \infty$ .

drawdown, and  $\rho_f$  is the first time  $X$  falls from its maximum  $\bar{X}$  by  $(100 \times \alpha)\%$ , as depicted in Figure 1 with  $\alpha = 0.3$ .

**Remark 2.2.** If floor functions  $f_1(\cdot), f_2(\cdot)$  both satisfy (2), and  $f_1(x) \leq f_2(x)$  for all  $x \in I$ , then for every fixed  $x \in I$ , we have the inequalities:

$$h(x) \leq v_{f_1}(x, \bar{x}) \leq v_{f_2}(x, \bar{x}) \leq \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} h(X_\tau) \mathbf{1}_{\{\tau < \infty\}}),$$

where  $\mathcal{T}$  is the set of all stopping times of  $X$ .

Given the optimal value  $v_f(x, \bar{x})$ , another related problem is

$$v_f^{(1)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-\hat{q}\tau} (v_f(X_\tau, X_\tau) - h(X_\tau) - c) \mathbf{1}_{\{\tau < \infty\}}), \quad (7)$$

where  $\hat{q} \in (0, q]$  is the discounting rate,  $c \in [0, \sup_{x \in I} (v_f(x, x) - h(x))]$  is the transaction fee,<sup>2</sup>  $\mathcal{T}$  is the set of all stopping times w.r.t. the filtration generated by  $X$ . The problem arises, for example, in optimal acquisition of the asset  $X$  when  $h(x) = x$ . In general, if we assume that  $h(X)$  is the price the investor need to pay to acquire one unit of the risky asset, then (7) represents the problem of finding the optimal time to purchase this risk asset. Note that the investor will select the optimal time to sell but subject to a trailing stop exit. For this reason, we will call the problem in (7) the optimal acquisition problem with a trailing stop, even for a general reward  $h(\cdot)$ .

**Remark 2.3.** Note that in (7), we apply the value function  $v_f(x, \bar{x})$  only with  $x = \bar{x}$ . From a practical point of view, this is the most relevant case since a trailing stop should be placed based on the price at which the asset was purchased, rather than an arbitrary reference price.

In summary, the solutions to (7) and (4) yield the optimal trading strategy that involves buying a risky asset and selling it later while being protected by a trailing stop.

## 2.1 Preliminaries of linear diffusions

It is well known that, for any  $q > 0$ , the *Sturm-Liouville equation*  $(\mathcal{L} - q)u(x) = 0$  has a positive increasing solution  $\phi_q^+(\cdot)$  and a positive decreasing solution  $\phi_q^-(\cdot)$ . In fact, for an arbitrary fixed  $\kappa \in \mathbb{R}_+$ , the solutions can be expressed as

$$\phi_q^+(x) = \begin{cases} \mathbb{E}_x(e^{-q\tau_X^+(\kappa)}), & \text{if } x \leq \kappa \\ \frac{1}{\mathbb{E}_\kappa(e^{-q\tau_X^+(x)})}, & \text{if } x > \kappa \end{cases}, \quad \phi_q^-(x) = \begin{cases} \frac{1}{\mathbb{E}_\kappa(e^{-q\tau_X^-(x)})}, & \text{if } x \leq \kappa \\ \mathbb{E}_x(e^{-q\tau_X^-(\kappa)}), & \text{if } x > \kappa \end{cases}, \quad \forall x \in I, \quad (8)$$

where  $\tau_X^+(y)$  and  $\tau_X^-(y)$  are the first passage times of  $X$  to level  $y \in I$  from below and above, respectively,

$$\tau_X^\pm(y) := \inf\{t > 0 : X_t \gtrless y\}, \quad \forall y \in I. \quad (9)$$

The functions  $\phi_q^\pm(\cdot)$  are also closely related to two-sided exit problems of  $X$ . Specifically, we have:

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<sup>2</sup>If  $c$  is too large then it is never optimal to exercise so the problem has 0 value.

**Lemma 2.1.** [Lehoczky (1977)] Suppose that  $l < y \leq x \leq z < r$ , then for  $q > 0$ , we have

$$\begin{aligned}\mathbb{E}_x(e^{-q\tau_X^-(y)} \mathbf{1}_{\{\tau_X^-(y) < \tau_X^+(z)\}}) &= \frac{\phi_q^-(x) \psi_q(z) - \psi_q(x)}{\phi_q^-(y) \psi_q(z) - \psi_q(y)}, \\ \mathbb{E}_x(e^{-q\tau_X^+(z)} \mathbf{1}_{\{\tau_X^-(y) > \tau_X^+(z)\}}) &= \frac{\phi_q^-(x) \psi_q(x) - \psi_q(y)}{\phi_q^-(z) \psi_q(z) - \psi_q(y)},\end{aligned}$$

where  $\psi_q : I \mapsto \mathbb{R}_+$  is a strictly increasing function defined as

$$\psi_q(x) := \frac{\phi_q^+(x)}{\phi_q^-(x)}, \quad \forall x \in I. \quad (10)$$

**Remark 2.4.** By the boundary behavior of  $X$ , we have  $\psi_q(l+) = 0$  and  $\psi_q(r-) = \infty$ .

## 2.2 Standing assumption

We now discuss the following standing assumption on the reward function  $h(\cdot)$ .

**Assumption 2.1.** Define the function

$$H(z) := \frac{h(x)}{\phi_q^-(x)}, \quad \text{where } z = \psi_q(x) \in \mathbb{R}_+. \quad (11)$$

We assume that:

- (i) There is  $z_0 \in \mathbb{R}_+$  such that  $H(\cdot)$  is convex over  $(0, z_0)$  and concave over  $[z_0, \infty)$ ;
- (ii) We have

$$H(0) := \lim_{z \downarrow 0} H(z) = 0, \quad \text{and} \quad \sup_{z > z_0} \frac{H(z)}{z} > H'_+(\infty), \quad (12)$$

where  $H'_+(z)$  is the right derivative of  $H(\cdot)$  at  $z$ .

**Remark 2.5.** Recall that  $\{x \in I : H(\psi_q(x)) > 0\} \equiv \{x \in I : h(x) > 0\} \neq \emptyset$ , so the convexity of  $H(\cdot)$  in Assumption 2.1 implies that there exists a  $z_1 \in [0, \infty)$  such that  $H(z) > 0$  for all  $z \in (z_1, \infty)$  and  $H(z) < 0$  for all  $z \in (0, z_1)$ . Moreover, we must have  $z_1 < z_0$  (otherwise  $H(z) < 0$  on  $\mathbb{R}_+$ ).

We remark that, when the reward function  $h(\cdot)$  is twice differentiable, Assumption 2.1 can be conveniently verified by the super/sub-harmonic property of  $h(\cdot)$ .

**Lemma 2.2.** Let the reward function  $h(\cdot)$  be twice differentiable and satisfy (12). Then, Assumption 2.1 holds if there exists a constant  $x_0 \in I$  such that  $(\mathcal{L} - q)h(x) \geq 0$  if and only if  $x \leq x_0$ .

*Proof.* The claim follows from Section 6 of Dayanik and Karatzas (2003). Indeed, we have

$$H''(z) = \frac{2}{\sigma^2(x)\phi_q^-(x)(\psi_q'(x))^2} ((\mathcal{L} - q)h(x)), \quad \text{for } z = \psi_q(x).$$

So  $H''(z) \geq 0$  if and only if  $z \leq \psi_q(x_0)$ . □

### 3 Optimal trading with a fixed stop-loss

To gain some intuition for our solution method for the problems in (4) and (7) with a trailing stop, we first consider the optimal stopping problems when the investor uses a fixed stop-loss exit instead of a trailing stop. Precisely, arbitrarily fix a  $y \in I$ , we consider the following class of problems indexed by  $y$ :

$$V_y(x) := \sup_{\tau \in \mathcal{T}_y^S} \mathbb{E}_x(e^{-q\tau} h(X_\tau) \mathbf{1}_{\{\tau < \infty\}}), \quad (13)$$

$$V_y^{(1)} := \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-q\tau} (V_y(X_\tau) - h(X_\tau) - c) \mathbf{1}_{\{\tau < \infty\}}), \quad (14)$$

where  $\mathcal{T}_y^S$  is the set of all stopping times of  $X$  that stops no later than the first passage time to level  $y$ , i.e.

$$\tau_X^-(y) = \inf\{t > 0 : X_t < y\}, \quad (15)$$

$\mathcal{T}$  is the set of all stopping times of  $X$ , and  $c \in [0, \sup_{x \in I} (V_y(x) - h(x))]$  is a transaction fee for asset acquisition. The problem in (13) puts a mandatory liquidation constraint upon hitting the fixed stop-loss level  $y$  from above.

The special cases of the problems in (13) and (14) with the reward function  $h(x) = x - c$  driven by the OU and CIR processes have been studied in Cartea et al. (2015); Leung and Li (2015); Leung et al. (2014, 2015). In this section, we present the analysis of problems (13) and (14) driven by a general linear diffusion.

#### 3.1 Optimal liquidation subject to a stop-loss exit

We now study the optimal liquidation problem (13) where  $X$  follows a general linear diffusion (see (1)). To facilitate our analysis, we also consider the extended case of (13) for  $y = l$ , in which case we have

$$V_l(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-q\tau} h(X_\tau) \mathbf{1}_{\{\tau < \infty\}}). \quad (16)$$

**Remark 3.1.** For any  $x \in I$ , the mapping  $y \mapsto V_y(x)$  is obviously non-increasing over  $[0, \infty)$ .

**Remark 3.2.** The connection between (4) and (13) can be seen as follows. For any  $x, \bar{x} \in I$  such that  $x \in (f(\bar{x}), \bar{x}]$ , by the  $\mathbb{P}_{x, \bar{x}}$ -a.s. inequality that  $\rho_f \leq \tau_X^-(f(\bar{x}))$ , we know that  $\mathcal{T}_f^T \subset \mathcal{T}_{f(\bar{x})}^S$ . Hence,  $v_f(x, \bar{x}) \leq V_{f(\bar{x})}(x)$ . As a consequence, if we define the optimal liquidation regions

$$\mathcal{S}_f^{T,L}(\bar{x}) := \{x \in (l, \bar{x}] : v_f(x, \bar{x}) = h(x)\}, \quad \forall \bar{x} \in I, \quad (17)$$

$$\mathcal{S}_y^{S,L} := \{x \in I : V_y(x) = h(x)\}, \quad \forall y \in I, \quad (18)$$

then we have

$$(\mathcal{S}_{f(\bar{x})}^{S,L} \cup (l, \bar{x}]) \subset \mathcal{S}_f^{T,L}(\bar{x}), \quad \forall \bar{x} > 0.$$

Additionally, if  $\bar{x} \in \mathcal{S}_{f(\bar{x})}^{S,L}$  then we have  $(\mathcal{S}_{f(\bar{x})}^{S,L} \cup (l, \bar{x})) = \mathcal{S}_f^{T,L}(\bar{x})$ , since in this case it is optimal to liquidate before  $X$  reaching a new maximum.

**Theorem 3.1.** Under Assumption 2.1, for any fixed  $y \in (l, \psi_q^{-1}(z_0))$ , there is a finite threshold  $b(y)$  such

that<sup>3</sup>

$$V_y(x) = \mathbb{E}_x(e^{-q(\tau_X^+(b(y)) \wedge \tau_X^-(y))} h(X_{\tau_X^+(b(y)) \wedge \tau_X^-(y)})), \quad \forall x \in I. \quad (19)$$

Moreover, the mapping  $y \mapsto b(y)$  is non-increasing over  $(l, \psi_q^{-1}(z_0))$ , with limits  $b(\psi_q^{-1}(z_0)-) = \psi_q^{-1}(z_0)$ , and  $b(l+) < r$ .

*Proof.* According to Dayanik and Karatzas (2003),  $V_y(\psi_q^{-1}(z))/\phi_q^-(\psi_q^{-1}(z))$  is the smallest concave majorant of  $H(z)$  for all  $z > \psi_q(y)$ , which we denote by  $\hat{H}(z)$ . By the convexity of  $H(\cdot)$ , we know this concave majorant is given by

$$\hat{H}_y(z) = \begin{cases} H(\psi_q(y)) \frac{z(y) - z}{z(y) - \psi_q(y)} + H(z(y)) \frac{z - \psi_q(y)}{z(y) - \psi_q(y)}, & \forall z \in (\psi_q(y), z(y)], \\ H(z), & \forall z \in (0, \psi_q(y)] \cup (z(y), \infty), \end{cases} \quad (20)$$

where  $z(y)$  is defined as

$$z(y) := \inf \arg \max_{z > z_0} \frac{H(z) - H(\psi_q(y))}{z - \psi_q(y)}. \quad (21)$$

Now define the barrier  $b(y) := \psi_q^{-1}(z(y))$ , we have  $\mathcal{S}_y^{\text{S,L}} = (l, y] \cup [b(y), r)$ . From Remark 3.1 we know that, for  $l \leq y_1 < y_2 < \psi_q^{-1}(z_0)$ , the equalities hold:

$$((l, y_1] \cup [b(y_1), r)) \equiv \mathcal{S}_{y_1}^{\text{S,L}} \subset \mathcal{S}_{y_2}^{\text{S,L}} \equiv ((l, y_2] \cup [b(y_2), r)).$$

Thus necessarily,  $b(y_2) \leq b(y_1)$ . To show the boundedness of  $b(y)$ , we consider the special case that  $y = l$ . In this case, define the function  $F(z) := H(z)/z$ , which is continuous by the convexity of  $H(\cdot)$ . We need to show that  $F(\cdot)$  attains its supremum over  $(z_0, \infty)$  at a finite point. Suppose this is not true, which means that  $F(z)$  must be maximized as  $z \rightarrow \infty$ . However, by Assumption 2.1, we find a sufficiently large  $z_1 > z_0$  such that

$$\sup_{z > z_0} F(z) > H'_+(z_1).$$

This implies that (using concavity of  $H(\cdot)$  at  $z_1$ )

$$\sup_{z > z_0} F(z) = \limsup_{z \rightarrow \infty} F(z) \leq \limsup_{\Delta \rightarrow \infty} \frac{H(z_1) + H'_+(z_1)\Delta}{z_1 + \Delta} = H'_+(z_1),$$

which is a contradiction to our choice of  $z_1$ . Hence we must have  $z(l) < \infty$  so  $b(l) < r$ , and  $b(y) \leq b(l+) \leq b(l) < r$  by the monotonicity.

As  $y \uparrow \psi_q^{-1}(z_0)$ ,  $b(y)$  converges to some limit in  $[\psi_q^{-1}(z_0), r)$ . Suppose  $b(\psi_q^{-1}(z_0)-) \equiv \underline{b} > \psi_q^{-1}(z_0)$ , then the concavity of  $H(\cdot)$  over  $(z_0, \infty)$  implies that

$$H'_+(\underline{b}) \geq \frac{H(\underline{b}) - H(z_0)}{\psi_q(\underline{b}) - z_0}.$$

However, by the definition of  $z(y)$ , this inequality is in fact an equality. This implies that  $H(\cdot)$  is in fact a straight line over  $[z_0, \psi_q^{-1}(\underline{b})]$ , but then (by the definition of  $z(y)$ , again) we must have  $b(\psi_q^{-1}(z_0)-) = \psi_q^{-1}(z_0)$  instead.  $\square$

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<sup>3</sup>Notice that in the expectation (19) the indicator  $\mathbf{1}_{\{\tau_X^+(b(y)) \wedge \tau_X^-(y) < \infty\}}$ , as it is equal to 1 almost surely.



**Remark 3.3.** The mapping  $y \mapsto b(y)$  is not necessarily continuous on  $(0, \psi_q^{-1}(z_0))$ . For example, if  $H(\cdot)$  is piecewise linear and concave on  $(z_0, \infty)$ , and we denote set of all kinks'  $x$ -coordinates by  $\mathcal{K}$ . Then  $b(y)$  may be a piecewise constant function that maps onto  $\mathcal{K}$ . In this case, the function  $b(y)$  is right continuous with a left limit at every  $y \in (l, \psi_q^{-1}(z_0)]$ .

The proof of Theorem 3.1 gives an effective way to calculate the optimal threshold  $b(y)$  when  $H(\cdot)$  (or equivalently  $h(\cdot)$ ) is continuously differentiable.

**Corollary 3.1.** In addition to Assumption 2.1, assume that  $h(\cdot)$  is continuously differentiable over  $(\psi_q^{-1}(z_0), r)$ , then for any  $y \in [l, \psi_q^{-1}(z_0))$ , the optimal threshold  $b(y)$  is the smallest solution over  $(\psi_q^{-1}(z_0), r)$  to

$$\frac{1}{\psi_q'(b)} \left( \frac{h'(b)}{\phi_q^-(b)} - \frac{h(b)\phi_q^{-\prime}(b)}{(\phi_q^-(b))^2} \right) = \frac{1}{\psi_q(b) - \psi_q(y)} \left( \frac{h(b)}{\phi_q^-(b)} - \frac{h(y)}{\phi_q^-(y)} \right).$$

**Remark 3.4.** A non-continuously differentiable reward function  $h(\cdot)$  may be chosen because one can still obtain a similar characterization of the optimal barrier  $z(y) = \psi_q^{-1}(b(y))$  using the left/right derivative of  $H(\cdot)$ . Indeed, one way to construct the smallest concave majorant of  $H(\cdot)$  over  $[\psi_q(y), \infty)$  is to tune the  $z$ -value in  $(z_0, \infty)$  so that the line segment connecting points  $(\psi_q(y), H(\psi_q(y)))$  and  $(z, H(z))$  does not go below the graph of  $H(\cdot)$ . But this line segments go below the graph of  $H(\cdot)$  if and only if

$$H'_-(z) < \frac{H(z) - H(\psi_q(y))}{z - \psi_q(y)}.$$

On the other hand, as a concave majorant, we automatically have that

$$H'_+(z(y)) \leq \frac{H(z(y)) - H(\psi_q(y))}{z(y) - \psi_q(y)}.$$

In summary, the optimal barrier  $z(y)$  is determined by

$$z(y) = \inf \{ z > z_0 : H'_+(z) \leq \frac{H(z) - H(\psi_q(y))}{z - \psi_q(y)} \leq H'_-(z) \}. \quad (22)$$

**Corollary 3.2.** If  $y \geq \psi_q^{-1}(z_0)$ , then the stopping region  $\mathcal{S}_y^{\text{S,L}} = I$ , i.e. there is no continuation region.

Next, we establish the monotonicity of the value function  $V_y(\cdot)$  over the continuation region for  $y \in (l, \psi_q^{-1}(z_0))$ .

**Proposition 3.1.** For any  $y \in (l, \psi_q^{-1}(z_0))$ , the value function  $V_y(\cdot)$  is strictly increasing over  $[y, b(y)]$ .

*Proof.* We begin by studying the limiting case of  $y = l$ . In the case  $y = l$ , the smallest concave  $\hat{H}_l(\cdot)$  in (20) must be strictly increasing over  $[0, z(l)]$ . Otherwise, we will have  $H(z) \leq \hat{H}_0(z) \leq 0$  for all  $z \in \mathbb{R}_+$ , which is contradiction to Remark 2.5. Therefore, we may write

$$\hat{H}_0(z) = \beta_1 z, \quad \forall z \in [0, z(l)],$$

for some  $\beta_1 > 0$ . But then it follows from Dayanik and Karatzas (2003) that

$$V_0(x) = \phi_q^-(x) \hat{H}_0(\psi_q(x)) = \beta_1 \phi_q^+(x), \quad \forall x \in (l, b(l)).$$

So the claim holds for the limiting case.

Now for any  $y \in (l, \psi_q^{-1}(z_0))$ , by (22) and the concavity of  $H(\cdot)$  over  $[z_0, \infty)$ , we know that

$$(\hat{H}_y)'_+(\psi_q(y)) = \frac{H(z(y)) - H(\psi_q(y))}{z(y) - \psi_q(y)} \text{ is increasing in } y.$$

Hence  $\hat{H}_y'(z) > \hat{H}_0'(z)$  for all  $z \in (\psi_q(y), z(y)) \subset (0, z(l))$ . Then by Section 6 of Dayanik and Karatzas (2003), this means that for all  $x \in (y, b(y)) \subset (l, b(l))$ ,

$$\frac{1}{\psi_q'(x)} \left( \frac{V_y'(x)}{\phi_q(x)} - V_y(x) \frac{\phi_q^{-\prime}(x)}{(\phi_q^-(x))^2} \right) = \hat{H}_y'(x) > \hat{H}_0'(x) = -\frac{1}{\psi_q'(x)} \left( \frac{V_0'(x)}{\phi_q(x)} - V_0(x) \frac{\phi_q^{-\prime}(x)}{(\phi_q^-(x))^2} \right).$$

Simplifying the above inequalities, we obtain

$$V_y'(x) - V_0'(x) > -\frac{\phi_q^{-\prime}(x)}{\phi_q^-(x)} (V_0(x) - V_y(x)).$$

Since  $\phi_q^-(\cdot)$  is positive and strictly decreasing, by Remark 3.1 we find that the right-hand side of the above inequality is nonnegative. As a consequence,  $V_y'(x) > V_0'(x) > 0$ , as claimed.  $\square$

If we define the early liquidation premium subject to the (fixed) stop-loss exit  $\tau_X^-(y)$  (see (15)) by the difference

$$P_y(x) := V_y(x) - \mathbb{E}_x(e^{-q\tau_X^-(y)} h(X_{\tau_X^-(y)}) \mathbf{1}_{\{\tau_X^-(y) < \infty\}}),$$

then we have the following characterizations.

**Corollary 3.3.** *We have*

1. *If  $l < y < \psi_q^{-1}(z_0)$  and  $y < x < b(y)$ , then*

$$P_y(x) = \phi_q^-(x) (H(\psi_q(b(y))) - H(\psi_q(y))) \frac{\psi_q(x) - \psi_q(y)}{\psi_q(b(y)) - \psi_q(y)}.$$

2. *If  $y < \psi_q^{-1}(z_0)$  and  $x \geq b(y)$  or  $y \geq \psi_q^{-1}(z_0)$ , then*

$$P_y(x) = h(x) - h(y) \frac{\phi_q^-(x)}{\phi_q^-(y)}.$$

*Proof.* Suppose that  $0 < y < \psi_q^{-1}(z_0)$  and  $y < x < b(y)$ . Then, the strong Markov property of  $X$  implies that

$$\begin{aligned} P_y(x) &= \mathbb{E}_x([h(b(y))e^{-q\tau_X^+(b(y))} - h(y)e^{-q\tau_X^-(y)}] \mathbf{1}_{\{\tau_X^+(b(y)) < \tau_X^-(y)\}}) \\ &= \mathbb{E}_x(e^{-q\tau_X^+(b(y))} [h(b(y)) - h(y)\mathbb{E}_{b(y)}(e^{-q\tau_X^-(y)})] \mathbf{1}_{\{\tau_X^+(b(y)) < \tau_X^-(y)\}}) \\ &= \left( h(b(y)) - h(y) \frac{\phi_q^-(b(y))}{\phi_q^-(y)} \right) \frac{\phi_q^-(x)}{\phi_q^-(b(y))} \frac{\psi_q(x) - \psi_q(y)}{\psi_q(b(y)) - \psi_q(y)} \\ &= \phi_q^-(x) (H(\psi_q(b(y))) - H(\psi_q(y))) \frac{\psi_q(x) - \psi_q(y)}{\psi_q(b(y)) - \psi_q(y)}, \end{aligned} \tag{23}$$

where we have used (8) and Lemma 2.1 in (23).

If  $l < y < \psi_q^{-1}(z_0)$  and  $x \geq b(y)$ , or  $y \geq \psi_q^{-1}(z_0)$ , then Theorem 3.1 and Corollary 3.2 imply that immediate stopping is optimal for problem (13). Hence, using (8) we arrive at

$$P_y(x) = h(x) - h(y) \mathbb{E}_x(e^{-q\tau_X^-} h(X_{\tau_X^-(y)}) \mathbf{1}_{\{\tau_X^-(y) < \infty\}}) = h(x) - h(y) \frac{\phi_q^-(x)}{\phi_q^-(y)}.$$

Hence, we conclude.  $\square$

### 3.2 Optimal acquisition

We now solve for the optimal acquisition timing corresponding to the optimal stopping problem

$$V_y^{(1)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-\hat{q}\tau} (V_y(X_\tau) - h(X_\tau) - c) \mathbf{1}_{\{\tau < \infty\}}), \quad (24)$$

where  $\mathcal{T}$  is the set of all stopping times of  $X$ . The associated optimal stopping region is

$$\mathcal{S}_y^{S,A} := \{x \in I : V_y^{(1)}(x) = V_y(x) - h(x) - c\}.$$

Without loss of generality, we may assume that

$$\sup_{x \in I} (V_y(x) - h(x)) > 0. \quad (25)$$

Otherwise, the value to (24) is 0. Given the known  $\psi_q$ -concavity of  $H(\cdot)$  and the value  $V_y(\cdot)$ , the problem in (24) can be conveniently solved if  $q = \hat{q}$  and  $c = 0$ . Indeed, by the construction in Theorem 3.1,

$$H_y^{(1)}(z) := \hat{H}_y(z) - H(z) = \begin{cases} 0, & \forall z \in \mathcal{S}_y^{S,L}, \\ H(\psi_q(y)) \frac{z(y) - z}{z(y) - \psi_q(y)} + H(z(y)) \frac{z - \psi_q(y)}{z(y) - \psi_q(y)} - H(z), & \forall z \in I \setminus \mathcal{S}_y^{S,L}. \end{cases}$$

So the convexity of  $H_y^{(1)}(z)$  on  $[\psi_q(y), z(y)]$  is completely determined by that of  $-H(z)$ , which is concave on  $(0, z_0)$  and is convex on  $[z_0, \infty)$  by Assumption 2.1. Moreover, by (25) we know that  $\sup_{z \in \mathbb{R}_+} H_y^{(1)}(z) > 0$  and at the maximum, the function is necessarily concave. Then it is easily seen that the optimal acquisition region:

$$\{x \in I : V_y^{(1)}(x) = V_y(x) - h(x)\} = [\psi_q^{-1}(\underline{z}^*(y)), \psi_q^{-1}(\bar{z}^*(y))],$$

where

$$\underline{z}^*(y) = \inf \arg \max_{z \in (\psi_q(y), z(y))} \frac{H_y^{(1)}(z)}{z}, \quad \bar{z}^*(y) = \sup \arg \max_{z \in \mathbb{R}_+} H_y^{(1)}(z). \quad (26)$$

**Theorem 3.2.** *If  $\hat{q} = q$  and  $c = 0$ , then under Assumption 2.1, we have  $\mathcal{S}_y^{S,A} = [\psi_q^{-1}(\underline{z}^*(y)), \psi_q^{-1}(\bar{z}^*(y))]$ , where  $\underline{z}^*(y)$  and  $\bar{z}^*(y)$  are given in (26). That is,*

$$V_y^{(1)}(x) = \mathbb{E}_x(e^{-q\theta_X(y)} (V_y(X_{\theta_X(y)}) - h(X_{\theta_X(y)})) \mathbf{1}_{\{\theta_X(y) < \infty\}}), \quad \forall x \in I,$$

where  $\theta_X(y) := \inf\{t > 0 : X_t \in \mathcal{S}_y^{S,A}\}$ .

In the general case that  $\hat{q} \in (0, q]$  and  $c \in [0, \sup_{x \in I} (V_y(x) - h(x))]$ , the analysis is more complicated but

the same idea goes through. More specially, we notice that

$$V_y(x) - h(x) - c = -c \leq 0, \quad \forall x \in \mathcal{S}_y^{\mathcal{S}, \mathcal{L}}.$$

So we must have  $\mathcal{S}_y^{\mathcal{S}, \mathcal{A}} \subset I \setminus \mathcal{S}_y^{\mathcal{S}, \mathcal{L}} = (y, b(y))$ . Hence, it will be essential to determine the  $\psi_{\hat{q}}$ -convexity of  $(V_y(x) - h(x) - c)/\phi_{\hat{q}}^-(x)$  in this interval. To help clarity of the results, we consider the following:

**Assumption 3.1.** *Define function*

$$K_y^{(1)}(z) := \frac{V_y(x) - h(x) - c}{\phi_{\hat{q}}(x)}, \quad \text{where } z = \psi_{\hat{q}}(x) \in \mathbb{R}_+.$$

We assume that there is a  $x_1 \in (y, b(y))$ , such that  $K_y^{(1)}(\cdot)$  is concave on  $(y, x_1)$ , and is convex on  $(x_1, b(y))$ .

**Remark 3.5.** *If the reward  $h(\cdot)$  is twice continuously differentiable, then Assumption 3.1 holds if  $(\mathcal{L} - \hat{q})h(x)$  is non-increasing on  $I$ . Indeed, recall that  $V_y(\cdot)$  is twice continuously differentiable over  $(y, b(y))$ , so  $(\mathcal{L} - \hat{q})V_y(x) = 0$ . As a result, the function*

$$(\mathcal{L} - \hat{q})(V_y(x) - h(x) - c) = (q - \hat{q})V_y(x) + \hat{q}c - (\mathcal{L} - \hat{q})h(x), \quad (27)$$

is non-decreasing in  $x$ , thanks to Proposition 3.1. In view of Lemma 2.2, Assumption 2.2 holds.

Under Assumption 3.1, the function  $K_y^{(1)}(\cdot)$  is concave on the left of its maximum over  $(y, b(y))$ . Thus, the optimal acquisition region must be

$$\{x \in I : V_y^{(1)}(x) = V_y(x) - h(x) - c\} = [\psi_{\hat{q}}^{-1}(\underline{z}_c^*(y)), \psi_{\hat{q}}^{-1}(\overline{z}_c^*(y))],$$

where

$$\underline{z}_c^*(y) = \inf \arg \max_{z \in (\psi_{\hat{q}}(y), \psi_{\hat{q}}(b(y)))} \frac{K_y^{(1)}(z) - \frac{c}{\phi_{\hat{q}}(l+)}}{z}, \quad \overline{z}_c^*(y) = \sup \arg \max_{z \in \mathbb{R}_+} K_y^{(1)}(z). \quad (28)$$

We now summarize our main result.

**Theorem 3.3.** *For  $\hat{q} \in (0, q]$  and  $c \in [0, \sup_{x \in I} (V_y(x) - h(x))]$ , under Assumption 2.1 and Assumption 3.1, we have*

$$\mathcal{S}_y^{\mathcal{S}, \mathcal{A}} = [\psi_{\hat{q}}^{-1}(\underline{z}_c^*(y)), \psi_{\hat{q}}^{-1}(\overline{z}_c^*(y))].$$

Therefore, the value function in (24) can be written as

$$V_y^{(1)}(x) = \mathbb{E}_x(e^{-q\theta_X^c(y)}(V_y(X_{\theta_X^c(y)}) - h(X_{\theta_X^c(y)}) - c)\mathbf{1}_{\{\theta_X^c(y) < \infty\}}), \quad \forall x \in I,$$

where  $\theta_X^c(y)$  is the first time  $X$  enters the region  $\mathcal{S}_y^{\mathcal{S}, \mathcal{A}}$ , defined by

$$\theta_X^c(y) := \inf\{t > 0 : X_t \in \mathcal{S}_y^{\mathcal{S}, \mathcal{A}}\}.$$

## 4 Optimal trading with a trailing stop

In this section, we apply the results we obtained to study the optimal liquidation problem (4) and the optimal acquisition problem (7).

## 4.1 Optimal liquidation

Returning to the problem in (4), we will first use results in Theorem 3.1 to construct a candidate threshold type strategy for liquidation before the trailing stop  $\rho_f$ .

**Corollary 4.1.** *There is a unique  $b_f^* \geq \psi_q^{-1}(z_0)$  such that  $b(f(\bar{x})) > \bar{x}$  if and only if  $\bar{x} < b_f^*$ . Moreover, we have  $f(b_f^*) \leq \psi_q^{-1}(z_0)$  and  $\Gamma(\bar{x}) > 0$  for all  $\bar{x} < b_f^*$ , where*

$$\Gamma(\bar{x}) := H'_+(\psi_q(\bar{x})) - \frac{H(\psi_q(\bar{x})) - H(\psi_q(f(\bar{x})))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))}. \quad (29)$$

*Proof.* From Theorem 3.1 we know that  $\bar{x} \mapsto b(f(\bar{x}))$  is non-increasing over  $(0, f^{-1}(\psi_q^{-1}(z_0)))$ , and the mapping  $\bar{x} \mapsto \bar{x}$  is strictly increasing over the same domain. Therefore, the ratio  $R(\bar{x}) := b(f(\bar{x}))/\bar{x}$  is strictly decreasing, and  $R(\bar{x}) \geq R(\psi_q^{-1}(z_0)) > 1$  for all  $\bar{x} < \psi_q^{-1}(z_0)$ , and by Theorem 3.1,

$$\lim_{\bar{x} \uparrow f^{-1}(\psi_q^{-1}(z_0))} R(\bar{x}) = \frac{\psi_q^{-1}(z_0)}{f^{-1}(\psi_q^{-1}(z_0))} < 1.$$

As a consequence, we have  $b_f^* = \inf\{\bar{x} < f^{-1}(\psi_q^{-1}(z_0)) : R(\bar{x}) \leq 1\}$ , and so  $f(b_f^*) \leq \psi_q^{-1}(z_0)$ .

Now for all  $\bar{x} < b_f^*$ , by the construction of  $b_f^*$  we have  $b(f(\bar{x})) > \bar{x}$ , by definition (see (22)) of  $z(f(\bar{x})) \equiv \psi_q(b(f(\bar{x}))) > \psi_q(\bar{x})$ , we know that

$$H'_+(\psi_q(\bar{x})) > \frac{H(\psi_q(\bar{x})) - H(\psi_q(f(\bar{x})))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} \Leftrightarrow \Gamma(\bar{x}) > 0.$$

This completes the proof.  $\square$

**Corollary 4.2.** *In addition to Assumption 2.1, assume that  $h(\cdot)$  is continuously differentiable over  $(\psi_q^{-1}(z_0), r)$ , then for the optimal threshold  $b_f^*$  is the unique solution over  $(\psi_q^{-1}(z_0), b(l))$  to*

$$\frac{1}{\psi'_q(b)} \left( \frac{h'(b)}{\phi_q^-(b)} - \frac{h(b)\phi_q^{-\prime}(b)}{(\phi_q^-(b))^2} \right) = \frac{1}{\psi_q(b) - \psi_q(f(b))} \left( \frac{h(b)}{\phi_q^-(b)} - \frac{h(f(b))}{\phi_q^-(f(b))} \right). \quad (30)$$

Let us suppose for now that  $\bar{x} \geq b_f^*$ . Then,

1. If we still have  $f(\bar{x}) < \psi_q^{-1}(z_0)$ , then by the definition of  $b_f^*$  given in Corollary 4.1, we have  $b(f(\bar{x})) \leq \bar{x}$ .

Thus, by Remark 3.2,

$$h(x) \leq u_f(x, \bar{x}) \leq V_{f(\bar{x})}(x), \quad \forall x, \bar{x} \in I \text{ with } x \leq \bar{x},$$

$$((l, f(\bar{x})) \cup [b(f(\bar{x})), \bar{x}]) \equiv (\mathcal{S}_{f(\bar{x})}^{\text{S,L}} \cup (l, \bar{x}]) = \mathcal{S}_f^{\text{T,L}}(\bar{x}).$$

2. If  $f(\bar{x}) \geq \psi_q^{-1}(z_0)$ , then by Corollary 3.2, we can use the same argument as above to conclude that  $(l, \bar{x}) \equiv (\mathcal{S}_{f(\bar{x})}^{\text{S,L}} \cup (l, \bar{x})) = \mathcal{S}_f^{\text{T,L}}(\bar{x})$ .

As a consequence we obtain the following theorem:

**Theorem 4.1.** *Under Assumption 2.1, for  $x, \bar{x} \in I$  with  $x \leq \bar{x}$  and  $\bar{x} \geq b_f^*$ , we have*

$$v_f(x, \bar{x}) \equiv V_{f(\bar{x})}(x).$$

In what follows we consider the remaining case  $l < x \leq \bar{x} < b_f^*$  and we shall establish the optimality of the stopping rule  $\tau_X^+(b_f^*) \wedge \rho_f$ . To this end, we first calculate the associated value of this strategy, denoted by  $u_f(x, \bar{x})$ . In particular, by the strong Markov property of  $X$ , applying Lemma 2.1 we have for any  $x \in (f(\bar{x}), \bar{x})$  with  $\bar{x} < b_f^*$ ,

$$\begin{aligned} u_f(x, \bar{x}) &= h(f(\bar{x})) \mathbb{E}_x(e^{-q\tau_X^-(f(\bar{x}))} \mathbf{1}_{\{\tau_X^-(f(\bar{x})) < \tau_X^+(\bar{x})\}}) + u_f(\bar{x}, \bar{x}) \mathbb{E}_x(e^{-q\tau_X^+(\bar{x})} \mathbf{1}_{\{\tau_X^+(\bar{x}) < \tau_X^-(f(\bar{x}))\}}) \\ &= \phi_q^-(x) \left( \frac{h(f(\bar{x}))}{\phi_q^-(f(\bar{x}))} \frac{\psi_q(\bar{x}) - \psi_q(x)}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} + \frac{u_f(\bar{x}, \bar{x})}{\phi_q^-(\bar{x})} \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} \right) \\ &= \phi_q^-(x) \left( H(\psi_q(f(\bar{x}))) \frac{\psi_q(\bar{x}) - \psi_q(x)}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} + H_f(\psi_q(\bar{x})) \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} \right), \end{aligned} \quad (31)$$

where for  $\bar{x} < b_f^*$ , we have defined

$$\begin{aligned} H_f(\psi_q(\bar{x})) &:= \frac{u_f(\bar{x}, \bar{x})}{\phi_q^-(\bar{x})} \\ &= \frac{h(b_f^*)}{\phi_q^-(\bar{x})} \mathbb{E}_{\bar{x}, \bar{x}}(e^{-q\tau_X^+(b_f^*)} \mathbf{1}_{\{\tau_X^+(b_f^*) < \rho_f\}}) + \frac{1}{\phi_q^-(\bar{x})} \mathbb{E}_{\bar{x}, \bar{x}}(e^{-q\rho_f} h(X_{\rho_f}) \mathbf{1}_{\{\rho_f < \tau_X^+(b_f^*)\}}). \end{aligned} \quad (32)$$

The two expectations in (32) can be computed using standard calculation using excursion theory:

**Proposition 4.1.** *For any  $b > \bar{x}$ , we have*

$$\mathbb{E}_{\bar{x}, \bar{x}}(e^{-q\rho_f} h(X_{\rho_f}) \mathbf{1}_{\{\rho_f < \tau_X^+(b)\}}) = \phi_q^-(\bar{x}) \int_{\bar{x}}^b \frac{h(f(v))}{\phi_q^-(f(v))} \frac{\psi_q'(v)}{\psi_q(v) - \psi_q(f(v))} \exp\left(-\int_{\bar{x}}^v \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right) dv,$$

and

$$\mathbb{E}_{\bar{x}, \bar{x}}(e^{-q\tau_X^+(b)} \mathbf{1}_{\{\tau_X^+(b) < \rho_f\}}) = \frac{\phi_q^-(\bar{x})}{\phi_q^-(b)} \exp\left(-\int_{\bar{x}}^b \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right).$$

In particular, as  $b \rightarrow r$  we obtain the value of the plain trailing stop (defined in (6))

$$g_f(\bar{x}, \bar{x}) = \phi_q^-(\bar{x}) \int_{\bar{x}}^r \frac{h(f(v))}{\phi_q^-(f(v))} \frac{\psi_q'(v)}{\psi_q(v) - \psi_q(f(v))} \exp\left(-\int_{\bar{x}}^v \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right) dv,$$

and for  $f(\bar{x}) < x \leq \bar{x}$ ,

$$g_f(x, \bar{x}) = \phi_q^-(x) \left( \frac{h(f(\bar{x}))}{\phi_q^-(f(\bar{x}))} \frac{\psi_q(\bar{x}) - \psi_q(x)}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} + \frac{g_f(\bar{x}, \bar{x})}{\phi_q^-(\bar{x})} \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} \right).$$

*Proof.* Let us denote by  $\mathbf{e}_q$  an exponential random variable with mean  $1/q$ , which is independent of  $X$ . Then we notice that

$$\begin{aligned} \mathbb{E}_{\bar{x}, \bar{x}}(e^{-q\rho_f} h(X_{\rho_f}) \mathbf{1}_{\{\tau_X^+(b) < \rho_f\}}) &= \mathbb{E}_{\bar{x}, \bar{x}}(h(X_{\rho_f}) \mathbf{1}_{\{\rho_f < \tau_X^+(b) \wedge \mathbf{e}_q\}}), \\ \mathbb{E}_{\bar{x}, \bar{x}}(e^{-q\tau_X^+(b)} \mathbf{1}_{\{\tau_X^+(b) < \rho_f\}}) &= \mathbb{P}_{\bar{x}, \bar{x}}(\tau_X^+(b) < \rho_f \wedge \mathbf{e}_q). \end{aligned}$$

To calculate the right-hand sides of the above, we consider an excursion of  $X$  below  $u$  (notice that  $\tau_X^+(u-) = \inf\{t > 0 : X_t \geq u\}$  is the first hitting time of  $X$  to  $u$ ):

$$\epsilon_u = \{\epsilon_u(s) := X_{\tau_X^+(u-)+s} - X_{\tau_X^+(u-)}\}_{0 < s \leq \tau_X^+(u) - \tau_X^+(u-)},$$

which is defined for all  $u \geq X_0 = \overline{X}_0 = \bar{x}$  such that its lifetime  $\zeta(\epsilon_u) := \tau_X^+(u) - \tau_X^+(u-) > 0$ . When  $\zeta(\epsilon_u) = 0$  we set  $\epsilon_u = \partial$ , an isolated point. Then the process  $\{(u, \epsilon_u)\}_{u \geq \bar{x}}$  is a Poisson point process with jump measure  $du \times dn_u$ , where  $n_u$  is the excursion measure for  $\epsilon_u$ . Define  $T_f(\epsilon_u) := \inf\{0 < s < \zeta(\epsilon_u) : \epsilon_u(s) > u - f(u)\}$ . It is known from Salminen et al. (2007) and Lemma 2.1 that,

$$\begin{aligned} n_u(\mathbf{e}_q < \zeta(\epsilon_u) \wedge T_f(\epsilon_u)) &= \lim_{x \uparrow u} \frac{1}{u - x} \left( 1 - \mathbb{E}_x(e^{-q\tau_X^-(f(u))} \mathbf{1}_{\{\tau_X^-(f(u)) < \tau_X^+(u)\}}) - \mathbb{E}_x(e^{-q\tau_X^+(u)} \mathbf{1}_{\{\tau_X^+(u) < \tau_X^-(f(u))\}}) \right) \\ &= \frac{\phi_q^{-, \prime}(u)}{\phi_q^-(u)} + \left( 1 - \frac{\phi_q^-(u)}{\phi_q^-(f(u))} \right) \frac{\psi_q'(u)}{\psi_q(u) - \psi_q(f(u))}, \end{aligned}$$

$$n_u(T_f(\epsilon_u) < \zeta(\epsilon_u) \wedge \mathbf{e}_q) = \lim_{x \uparrow u} \frac{\mathbb{E}_x(e^{-q\tau_X^-(f(u))} \mathbf{1}_{\{\tau_X^-(f(u)) < \tau_X^+(u)\}})}{u - x} = \frac{\phi_q^-(u)}{\phi_q^-(f(u))} \frac{\psi_q'(u)}{\psi_q(u) - \psi_q(f(u))}.$$

Hence,

$$n_u(\mathbf{e}_q < \zeta(\epsilon_u) \wedge T_f(\epsilon_u) \text{ or } T_f(\epsilon_u) < \zeta(\epsilon_u) \wedge \mathbf{e}_q) = \frac{\phi_q^{-, \prime}(u)}{\phi_q^-(u)} - \frac{\psi_q'(u)}{\psi_q(u) - \psi_q(f(u))}.$$

Let  $A$  be the space of all excursions  $\epsilon_u$  such that  $T_f(\epsilon_u) < \zeta(\epsilon_u) \wedge \mathbf{e}_q$ , and  $B$  be the space of all excursions  $\epsilon_u$  such that  $\mathbf{e}_q < \zeta(\epsilon_u) \wedge T_f(\epsilon_u)$ . We have that  $A \cap B = \emptyset$ . Consider a Poisson process (with time indexed by the running maximum  $\overline{X}$ ) that jumps whenever the current excursion  $\epsilon_{\overline{X}} \in A \cup B$ , then from the above calculation, we know that this Poisson process has jump intensity  $n_u(\mathbf{e}_q < \zeta(\epsilon_u) \wedge T_f(\epsilon_u) \text{ or } T_f(\epsilon_u) < \zeta(\epsilon_u) \wedge \mathbf{e}_q)$ . So  $\mathbb{P}_{\bar{x}, \bar{x}}(\tau_X^+(b) < \rho_f \wedge \mathbf{e}_q)$  is the same as the probability that this Poisson process has no jump over  $[\bar{x}, b)$ , which is given by

$$\exp\left(-\int_{\bar{x}}^b n_u(\mathbf{e}_q < \zeta(\epsilon_u) \wedge T_f(\epsilon_u) \text{ or } T_f(\epsilon_u) < \zeta(\epsilon_u) \wedge \mathbf{e}_q) du\right) = \frac{\phi_q^-(\bar{x})}{\phi_q^-(b)} \exp\left(-\int_{\bar{x}}^b \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right).$$

Moreover, for any  $v \in [\bar{x}, b)$ , the probability that the Poisson process will have the first jump at “time”  $dv$  as a result of  $\epsilon_v \in A$ , is given by

$$\begin{aligned} &\exp\left(-\int_{\bar{x}}^v n_u(\mathbf{e}_q < \zeta(\epsilon_u) \wedge T_f(\epsilon_u) \text{ or } T_f(\epsilon_u) < \zeta(\epsilon_u) \wedge \mathbf{e}_q) du\right) \times n_v(T_f(\epsilon_v) < \zeta(\epsilon_v) \wedge \mathbf{e}_q) dv \\ &= \frac{\phi_q^-(\bar{x})}{\phi_q^-(v)} \exp\left(-\int_{\bar{x}}^v \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right) \times \frac{\phi_q^-(v)}{\phi_q^-(f(v))} \frac{\psi_q'(v)}{\psi_q(v) - \psi_q(f(v))} dv \\ &= \frac{\phi_q^-(\bar{x})}{\phi_q^-(f(v))} \frac{\psi_q'(v)}{\psi_q(v) - \psi_q(f(v))} \exp\left(-\int_{\bar{x}}^v \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right) dv, \end{aligned}$$

which is the same as  $\mathbb{P}_{\bar{x}, \bar{x}}(\overline{X}_{\rho_f} \in dv, \rho_f < \tau_X^+(b) \wedge \mathbf{e}_q)$ . The proof is complete by integrating in  $v$  over  $[\bar{x}, b)$ .  $\square$

To establish the optimality of  $\tau_X^+(b_f^*) \wedge \rho_f$  when  $0 < x \leq \bar{x} < b_f^*$ , we need to show that the value of the rule  $u_f(x, \bar{x})$  dominates the reward function  $h(x)$ . In what follows we shall prove this fact by using (31) and the optimality of  $b_f^*$  (see Corollary 4.1).

**Lemma 4.1.** *For all  $\bar{x} \in (l, b_f^*)$ , we have  $H_f(\psi_q(\bar{x})) > H(\psi_q(\bar{x}))$ .*

*Proof.* Let us define for any  $b \geq \bar{x}$

$$H_f(\psi_q(\bar{x}), b) = H(\psi_q(b)) \exp\left(-\int_{\bar{x}}^b \frac{\psi'_q(u) du}{\psi_q(u) - \psi_q(f(u))}\right) + \int_{\bar{x}}^b \frac{\psi'_q(v) H(\psi_q(f(v)))}{w(v) - w(f(v))} \exp\left(-\int_{\bar{x}}^v \frac{\psi'_q(u)}{\psi_q(u) - \psi_q(f(u))} du\right) dv.$$

It is clear that  $H_f(\psi_q(\bar{x}), \bar{x}) = H(\psi_q(\bar{x})) = \frac{H_f(\bar{x})}{\phi_r^-(\bar{x})}$ , and for  $b > \bar{x}$  we have the right derivative of  $H_f(\psi_q(\bar{x}), b)$  in  $b$ :

$$\frac{\partial}{\partial_+ b} H_f(\psi_q(\bar{x}), b) = \psi'_q(b) \exp\left(-\int_{\bar{x}}^b \frac{\psi'_q(u) du}{\psi_q(u) - \psi_q(f(u))}\right) \left( H'_+(\psi_q(b)) - \frac{H(\psi_q(b)) - H(\psi_q(f(b)))}{\psi_q(b) - \psi_q(f(b))} \right).$$

It follows that the sign of  $\frac{\partial}{\partial_+ b} H_f(\psi_q(\bar{x}), b)$  depends on that of

$$\Gamma(b) = H'_+(\psi_q(b)) - \frac{H(\psi_q(b)) - H(\psi_q(f(b)))}{\psi_q(b) - \psi_q(f(b))}.$$

But the latter is known to be positive for all  $b < b_f^*$ , thanks to Corollary 4.1. Because  $H'_+(\psi_q(\cdot))$  is continuous almost everywhere, so is  $\Gamma(\cdot)$ . So we know that

$$H_f(\psi_q(\bar{x})) \equiv H_f(\psi_q(\bar{x}), b_f^*) = H(\psi_q(\bar{x})) + \int_{\bar{x}}^{b_f^*} \frac{\partial}{\partial_+ u} H_f(\psi_q(\bar{x}), u) du > H(\psi_q(\bar{x})), \forall \bar{x} < b_f^*.$$

This completes the proof.  $\square$

Notice that Lemma 4.1 and (31) imply that the value of the strategy  $\tau_X^+(b_f^*) \wedge \rho_f$  satisfies inequality

$$\frac{u_f(x, \bar{x})}{\phi_q^-(x)} > H(\psi_q(f(\bar{x}))) \frac{\psi_q(\bar{x}) - \psi_q(x)}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} + H(\psi_q(\bar{x})) \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))}, \quad (33)$$

for all  $x \in (f(\bar{x}), \bar{x})$  and  $\bar{x} < b_f^*$ . In the sequel we will prove using (33) that  $u_f(x, \bar{x}) > h(x)$  for all  $x \in (f(\bar{x}), \bar{x})$ .

1. If  $\bar{x} < \psi_q^{-1}(z_0)$ , then by the convexity of  $H(\cdot)$  over  $(0, z_0)$  we know from (33) that

$$\frac{u_f(x, \bar{x})}{\phi_q^-(x)} > H(\psi_q(x)) = \frac{h(x)}{\phi_q^-(x)}, \quad (34)$$

which implies that  $u_f(x, \bar{x}) > h(x)$  for  $x \in (f(\bar{x}), \bar{x})$ .

2. If  $\psi_q^{-1}(z_0) \leq \bar{x} < b_f^*$ , we claim that the above inequality also holds, i.e. the straight line segments connecting points  $(\psi_q(f(\bar{x})), H(\psi_q(f(\bar{x}))))$  and  $(\psi_q(\bar{x}), H(\psi_q(\bar{x})))$  stays above  $H(\cdot)$  on the interval  $(\psi_q(f(\bar{x})), \psi_q(\bar{x}))$ . Suppose not, then since  $H(\cdot)$  is convex at  $\psi_q(f(\bar{x}))$  (we know from Corollary 4.1 that  $f(b)_f^* \leq \psi_q^{-1}(z_0)$ ), the only possibility for  $H(\cdot)$  to touch the above line segment on interval  $(\psi_q(f(\bar{x})), \psi_q(\bar{x}))$  is that  $H(\cdot)$  changes convexity at least twice, for otherwise there is no way for the line segments to have intercepts with  $H(\cdot)$  other than the two boundaries. However, as assumed  $H(\cdot)$  changes convexity exactly once at  $z_0$ , so the line segments must stay above  $H(\cdot)$  over  $(\psi_q(f(\bar{x})), \psi_q(\bar{x}))$ . That is, inequality (34) holds.



Overall, we have shown that

$$\{e^{-q(t \wedge \tau_X^+(b_f^*) \wedge \rho_f)} u_f(X_{t \wedge \tau_X^+(b_f^*) \wedge \rho_f}, \bar{X}_{t \wedge \tau_X^+(b_f^*) \wedge \rho_f})\}_{t \geq 0} \text{ is a } \mathbb{P}_{x, \bar{x}}\text{-martingale.}$$

$$u_f(x, \bar{x}) > h(x), \quad \forall x, \bar{x} \in \mathbb{R}_+ \text{ such that } f(\bar{x}) < x \leq \bar{x} < b_f^*.$$

Because waiting until  $\tau_X^+(b_f^*) \wedge \rho_f$  yields positive “time value”  $u_f(x, \bar{x}) - h(x) > 0$  for all  $f(\bar{x}) < x \leq \bar{x} < b_f^*$ , we know that this region should be part of the optimal stopping region. The last step to establish the optimality of  $\tau_X^+(b_f^*) \wedge \rho_f$  is to verify that it is not optimal to not to stop after  $\tau_X^+(b_f^*)$  on the event  $\{\tau_X^+(b_f^*) < \rho_f\}$ . In other words, we need to show that, once  $\bar{X}$  reaches  $b_f^*$ , it will not be optimal to be “greedy” and wait for other potentially higher liquidation prices. Indeed, this has already been proved in Theorem 4.1.

In summary, we obtain the following result:

**Theorem 4.2.** *Under Assumption 2.1, we have for all  $l < x \leq \bar{x} < b_f^*$  that,*

$$v_f(x, \bar{x}) \equiv u_f(x, \bar{x}) = \mathbb{E}_{x, \bar{x}}(e^{-q(\tau_X^+(b_f^*) \wedge \rho_f)}) h(X_{\tau_X^+(b_f^*) \wedge \rho_f}),$$

where  $b_f^*$  is defined in Corollary 4.1. Moreover, the mapping  $f \mapsto b_f^*$  is non-decreasing over all functions satisfying (2).

*Proof.* The only claim that needs a proof is the monotonicity of  $f \mapsto b_f^*$ . But that is due to Remark 2.2 and the structure of the optimal stopping region.  $\square$

**Corollary 4.3.** *The value of the plain trailing stop  $g_f(x, \bar{x})$  given in Proposition 4.1 is finite. Moreover, for any  $f(\bar{x}) < x \leq \bar{x} < b_f^*$ , the early liquidation premium given the trailing stop  $\rho_f$  is given by*

$$p_f(x, \bar{x}) = \frac{\phi_q^-(x)}{\phi_q^-(b_f^*)} \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} \exp\left(-\int_{\bar{x}}^{b_f^*} \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right) (h(b_f^*) - g_f(b_f^*, b_f^*)),$$

where  $g_f(b_f^*, b_f^*)$  is given in Proposition 4.1. If  $f(\bar{x}) < \psi_q^{-1}(z_0)$ ,  $\bar{x} \geq b_f^*$  and  $f(\bar{x}) < x < b(f(\bar{x}))$  (see Theorem 3.1 for the existence of  $b(y)$ ), then the early liquidation premium given the trailing stop  $\rho_f$  is

$$p_f(x, \bar{x}) = \frac{\phi_q^-(x)}{\phi_q^-(b(f(\bar{x})))} \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi(b(f(\bar{x}))) - \psi_q(f(\bar{x}))} (h(b(f(\bar{x}))) - g_f(b(f(\bar{x})), \bar{x}).$$

Finally, if  $f(\bar{x}) < \psi_q^{-1}(z_0)$ ,  $\bar{x} \geq b_f^*$  and  $b(f(\bar{x})) \leq x \leq \bar{x}$ , or  $f(\bar{x}) \geq \psi_q^{-1}(z_0)$  and  $f(\bar{x}) < x \leq \bar{x}$ , then the early liquidation premium given the trailing stop  $\rho_f$  is

$$p_f(x, \bar{x}) = h(x) - g_f(x, \bar{x}).$$

*Proof.* If  $f(\bar{x}) < x \leq \bar{x} < b_f^*$ , then by the strong Markov property of  $X$ , we have

$$\begin{aligned} p_f(x, \bar{x}) &= \mathbb{E}_{x, \bar{x}}([e^{-q\tau_X^+(b_f^*)} h(X_{\tau_X^+(b_f^*)}) - e^{-q\rho_f} h(X_{\rho_f})] \mathbf{1}_{\{\tau_X^+(b_f^*) < \rho_f < \infty\}}) \\ &= \mathbb{E}_{x, \bar{x}}(e^{-q\tau_X^+(b_f^*)} \mathbf{1}_{\{\tau_X^+(b_f^*) < \rho_f\}}) \left( h(b_f^*) - \mathbb{E}_{b_f^*, b_f^*}(e^{-q\rho_f} h(X_{\rho_f}) \mathbf{1}_{\{\rho_f < \infty\}}) \right), \end{aligned}$$

where  $\mathbb{E}_{b_f^*, b_f^*}(e^{-q\rho_f} h(X_{\rho_f}) \mathbf{1}_{\{\rho_f < \infty\}}) = g_f(b_f^*, b_f^*)$  is given in Proposition 4.1, which is finite since we know that it is dominated from above by  $v_f(b_f^*, b_f^*) = h(b_f^*)$ . On the other hand, by the analysis in (31) and the

results in Proposition 4.1, we have

$$\mathbb{E}_{x,\bar{x}}(e^{-q\tau_X^+(b_f^*)}\mathbf{1}_{\{\tau_X^+(b_f^*) < \rho_f\}}) = \frac{\phi_q^-(x)}{\phi_q^-(b_f^*)} \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi_q(\bar{x}) - \psi_q(f(\bar{x}))} \exp(-\int_{\bar{x}}^{b_f^*} \frac{\psi_q'(u)du}{\psi_q(u) - \psi_q(f(u))}).$$

We obtain the claimed formula by combining the above results.

If  $f(\bar{x}) < \psi_q^{-1}(z_0)$  and  $\bar{x} \geq b_f^*$ , then from Theorem 3.1 and Theorem 4.1 we know that  $b(f(\bar{x})) \leq \bar{x}$ , and for all  $f(\bar{x}) < x < b(f(\bar{x}))$ ,

$$\begin{aligned} & p_f(x, \bar{x}) \\ &= \mathbb{E}_x(e^{-q\tau_X^+(b(f(\bar{x})))}\mathbf{1}_{\{\tau_X^+(b(f(\bar{x}))) < \tau_X^-(f(\bar{x}))\}}) (h(b(f(\bar{x}))) - \mathbb{E}_{b(f(\bar{x}), \bar{x})}(e^{-q\rho_f}h(X_{\rho_f})\mathbf{1}_{\{\rho_f < \infty\}})) . \end{aligned}$$

By using Lemma 2.1 we obtain that

$$\mathbb{E}_x(e^{-q\tau_X^+(b(f(\bar{x})))}\mathbf{1}_{\{\tau_X^+(b(f(\bar{x}))) < \tau_X^-(f(\bar{x}))\}}) = \frac{\phi_q^-(x)}{\phi_q^-(b(f(\bar{x})))} \frac{\psi_q(x) - \psi_q(f(\bar{x}))}{\psi(b(f(\bar{x}))) - \psi_q(f(\bar{x}))}.$$

The claim in this case follows from Proposition 4.1.

In the last case that  $f(\bar{x}) < \psi_q^{-1}(z_0)$ ,  $\bar{x} \geq b_f^*$  and  $b(f(\bar{x})) \leq x \leq \bar{x}$ , or  $f(\bar{x}) \geq \psi_q^{-1}(z_0)$  and  $f(\bar{x}) < x \leq \bar{x}$ , from Theorem 3.1 and Theorem 4.1 we know that the optimal stopping rule for problem (4) is 0, so we have

$$p_f(x, \bar{x}) = h(x) - \mathbb{E}_{x,\bar{x}}(e^{-q\rho_f}h(X_{\rho_f})\mathbf{1}_{\{\rho_f < \infty\}}).$$

This completes the proof.  $\square$

**Remark 4.1.** If the second condition in (12) fails to hold, then from the proofs of Theorem 3.1 and Corollary 4.1, we know that the optimal threshold  $b_f^*$  may be at the boundary  $r$ , in which case, it will be optimal not to liquidate before the trailing stop. That is,  $p_f(x, \bar{x}) = 0$  for all  $x, \bar{x} \in I$  such that  $x \in (f(\bar{x}), \bar{x}]$ .

## 4.2 Optimal acquisition with a trailing stop

In this section, we solve the optimal stopping problem related to acquisition with a trailing stop, which we recall as follows:

$$v_f^{(1)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-\hat{q}\tau}(v_f(X_\tau, X_\tau) - h(X_\tau) - c)\mathbf{1}_{\{\tau < \infty\}}), \quad (35)$$

where  $\mathcal{T}$  is the set of all stopping times of  $X$ , and  $c \in [0, \sup_{x \in I}(v_f(x, x) - h(x))]$ .

Let us define the optimal acquisition region with a trailing stop as

$$\mathcal{S}_f^{\text{T,A}} := \{x \in I : v_f^{(1)}(x) = v_f(x, x) - h(x) - c\}.$$

By Dayanik and Karatzas (2003) and (32), to determine  $\mathcal{S}_f^{\text{T,A}}$ , it suffices to obtain the smallest concave majorant of

$$H_{f,\hat{q}}^{(1)}(z) := \frac{v_f(x, x) - h(x) - c}{\phi_{\hat{q}}^-(x)}, \quad \text{where } z = \psi_{\hat{q}}(x) \in \mathbb{R}_+. \quad (36)$$

In light of Theorem 4.1, we know that for  $x \geq b_f^*$ , we have  $v_f(x, x) - h(x) - c = -c \leq 0$ , so we must have

$\mathcal{S}_f^{\top, \mathbf{A}} \subset I \setminus [b_f^*, r) = (l, b_f^*)$ . Therefore, if we denote by

$$\bar{z}_f^* := \sup \arg \max_{z \in \mathbb{R}_+} H_{f, \hat{q}}^{(1)}(z). \quad (37)$$

Then we have  $H_{f, \hat{q}}^{(1)}(\bar{z}_f^*) > 0$  (since  $c < \sup_{x>0} (v_f(x, x) - h(x))$ ), and  $\bar{z}_f^* \in [0, \psi_{\hat{q}}(b_f^*))$ , and the smallest concave majorant of  $H_{f, \hat{q}}^{(1)}(\cdot)$  over  $[\bar{z}_f^*, \infty)$  must be given by the constant function  $H_{f, \hat{q}}^{(1)}(\bar{z}_f^*)$ . To determine whether  $\mathcal{S}_f^{\top, \mathbf{A}}$  is one finite interval in  $(l, \psi_{\hat{q}}^{-1}(\bar{z}_f^*))$  as in Theorem 3.3 is quite a challenging task, due to lack of information about the convexity of function  $H_{f, \hat{q}}^{(1)}(\cdot)$ . But if it changes convexity only once over  $(0, \psi_{\hat{q}}(b_f^*))$ , we have the following results.

**Theorem 4.3.** *Assuming  $H_{f, \hat{q}}^{(1)}(\cdot)$  given in (36) changes from being concave to convex over  $(0, \psi_{\hat{q}}(b_f^*))$ , then under Assumption 2.1, we have  $\mathcal{S}_f^{\top, \mathbf{A}} = (l, \underline{b}_f^*]$ , where  $\underline{b}_f^* := \psi_{\hat{q}}^{-1}(\bar{z}_f^*)$  with  $\bar{z}_f^*$  given in (37). That is,*

$$v_f^{(1)}(x) = \mathbb{E}_x(e^{-\hat{q}\tau_{X^-}(\underline{b}_f^*)}(v_f(X_{\tau_{X^-}(\underline{b}_f^*)}, X_{\tau_{X^-}(\underline{b}_f^*)}) - h(X_{\tau_{X^-}(\underline{b}_f^*)}) - c)\mathbf{1}_{\{\tau_{X^-}(\underline{b}_f^*) < \infty\}}), \quad \forall x \in I.$$

We do not, however, give a simple, verifiable condition similar as in Remark 3.5 for Theorem 4.3, because the value function  $v_f(\cdot, \cdot)$  involves the floor function  $f(\cdot)$ . And even under the assumption that  $h(\cdot)$  is twice continuously differentiable, it is still very challenging to analyze the convexity of  $H_{f, \hat{q}}^{(1)}(\cdot)$ . Nonetheless, one can still numerically determine the optimal acquisition region. To fix the idea, let us assume for the moment that  $\hat{q} = q$ . Let us define

$$z_f^* := \psi_q(b_f^*), \quad \varphi(z) := \psi_q(f^{-1}(\psi_q^{-1}(z))), \quad \forall z \in \mathbb{R}_+. \quad (38)$$

It is clear that  $\varphi(\cdot)$  is an increasing function such that  $0 < \varphi(z) < z$ . Then from Proposition 4.1 we have for all  $z \in (0, z_f^*)$

$$H_f(z) = \exp(-\int_z^{z_f^*} \frac{d\nu}{\nu - \varphi(\nu)}) H(z_f^*) + \int_z^{z_f^*} H(\varphi(\nu)) \exp(-\int_z^\nu \frac{dw}{w - \varphi(w)}) \frac{d\nu}{\nu - \varphi(\nu)}, \quad \forall z \in (0, z_f^*). \quad (39)$$

To obtain the smallest concave majorant of  $H_{f, q}^{(1)}(z) = H_f(z) - H(z) - c/\phi_q^-(\psi_q^{-1}(z))$ , we need to numerically evaluate  $H_f(\cdot)$ . To that end, it will be more convenient to rewrite (39) into an equivalent first-order linear ODE form:

$$\begin{cases} H_f'(z) = \frac{H_f(z) - H(\varphi(z))}{z - \varphi(z)}, & \forall z \in (0, z_f^*), \\ \text{subject to } H_f(z_f^*) = H(z_f^*). \end{cases} \quad (40)$$

Then we can use Mathematica's NDSolve command to efficiently compute the values of  $H_{f, \hat{q}}^{(1)}(\cdot)$  and its derivatives.

The above method can also be applied to general cases that  $\hat{q} \in (0, q]$ . Indeed, define

$$H_{f, \hat{q}}(z) := \frac{v_f(x, x)}{\phi_{\hat{q}}^-(x)}, \quad \text{where } z = \psi_{\hat{q}}(x) \in \mathbb{R}_+.$$

Then by

$$\phi_{\hat{q}}^-(x) H_{f, \hat{q}}(\psi_{\hat{q}}(x)) = v_f(x, x) = \phi_q^-(x) H_f(\psi_q(x)), \quad \forall x \in I,$$

we obtain that

$$\phi_q^-(\psi_q^{-1}(z))H_{f,\hat{q}}(z) = v_f(\psi_q^{-1}(z), \psi_q^{-1}(z)) = \phi_q^-(\psi_q^{-1}(z))H_f(\psi_q(\psi_q^{-1}(z))).$$

Therefore,

$$H_{f,\hat{q}}(z) = \pi(z)H_f(\chi(z)), \quad (41)$$

where  $\pi(\cdot)$  and  $\chi(\cdot)$  are positive differentiable functions defined as

$$\pi(z) := \left( \frac{\phi_q^-}{\phi_{\hat{q}}^-} \right) \circ \psi_{\hat{q}}^{-1}(z), \quad \chi(z) := \psi_q \circ \psi_{\hat{q}}^{-1}(z).$$

By standard calculation, we obtain from (40) and (41) a first-order linear ODE for  $H_{f,\hat{q}}(\cdot)$ :

$$\begin{cases} H'_{f,\hat{q}}(z) = \left( \frac{\pi'(z)}{\pi(z)} + \frac{\chi'(z)}{\chi(z) - \varphi(\chi(z))} \right) H_{f,\hat{q}}(z) - \frac{\chi'(z)\pi(z)}{\chi(z) - \varphi(\chi(z))} H(\varphi(\chi(z))), & \forall z \in (0, \chi^{-1}(z_f^*)), \\ \text{subject to } H_{f,\hat{q}}(\chi^{-1}(z_f^*)) = \pi(\chi^{-1}(z_f^*))H(z_f^*). \end{cases} \quad (42)$$

## 5 Trading with a trailing stop under the exponential OU model

In this section, we apply the results in Section 4 to an exponential Ornstein-Uhlenbeck (OU) model:

$$dX_t = X_t \left( \lambda(\theta - \log X_t) + \frac{1}{2}\sigma^2 \right) dt + \sigma X_t dW_t, \quad X_0 = x \in I \equiv \mathbb{R}_+, \quad (43)$$

where  $W$  is a standard Brownian motion,  $\lambda, \sigma > 0$  are positive constants, and  $\theta \in \mathbb{R}$  is the long term average for the log-price  $\log X$ :

$$d(\log X_t) = \lambda(\theta - \log X_t)dt + \sigma dW_t.$$

With reference to (8), it is well-known (see p.542 of Borodin and Salminen (2002)) that

$$\phi_q^+(x) = e^{\frac{\lambda}{2\sigma^2}(y-\theta)^2} D_{-\frac{q}{\lambda}}\left(\frac{\sqrt{2\lambda}}{\sigma}(y-\theta)\right), \quad \phi_q^-(x) = e^{\frac{\lambda}{2\sigma^2}(y-\theta)^2} D_{-\frac{q}{\lambda}}\left(\frac{\sqrt{2\lambda}}{\sigma}(\theta-y)\right), \quad \text{where } y = \log x,$$

where  $D_\nu(\cdot)$  is the parabolic cylinder function with parameter  $\nu$ . We are interested in optimal liquidation and acquisition of one unit of a risky asset whose price is modeled by  $X$ . To that end, we let

$$h(x) = x - c_0, \quad \forall x \in I,$$

where  $c_0 \geq 0$  is a transaction cost to sell. Then it follows that, for any  $q > 0$

$$(\mathcal{L} - q)h(x) = \left( \lambda(\theta - \log x) + \frac{1}{2}\sigma^2 - q \right) x + qc_0, \quad \forall x \in I,$$

which is a strictly decreasing function with range equal to  $\mathbb{R}$ . Moreover, by the asymptotic behavior of  $D_\nu(\cdot)$  (see e.g. equation (1.8) of Temme (2000)), we know that the reward function  $h(\cdot)$  satisfies Assumption 2.1.

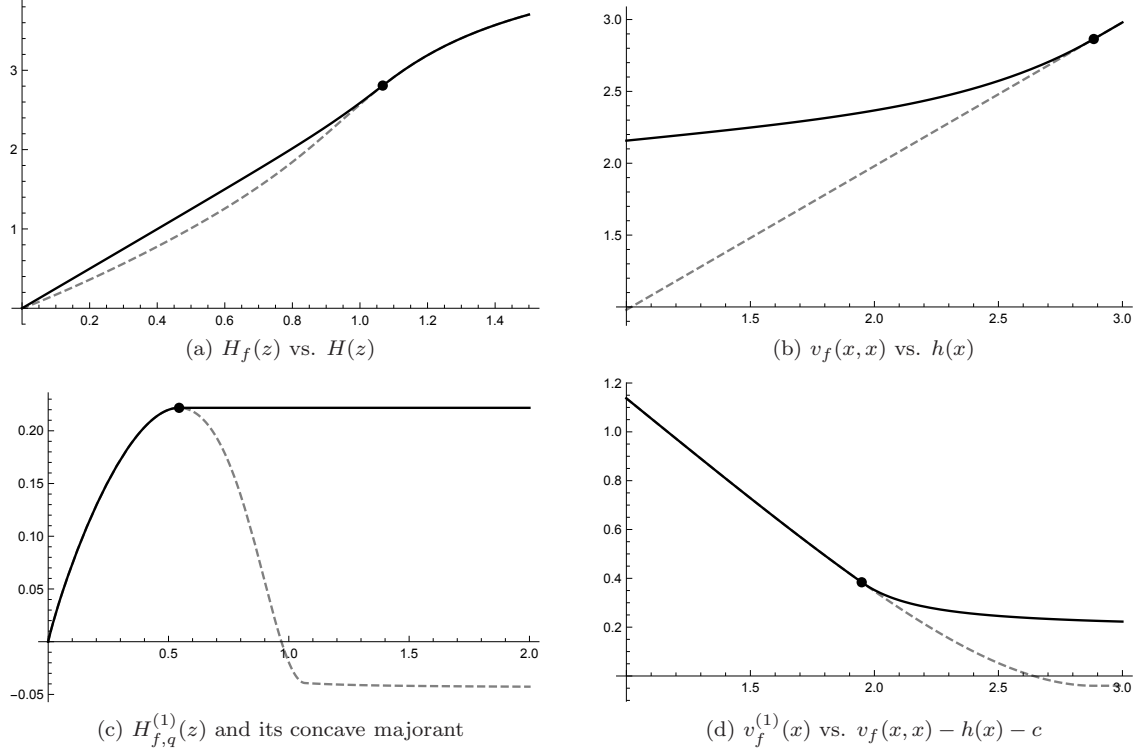


Figure 2: Numerical results under the exponential OU model (43): (a) Plots of function  $H(z)$  (dashed gray) and  $H_f(z)$  (solid black). The “pasting point”  $\psi_q(b_f^*) = 1.0674$  is indicated by the black dot. (b) Plots of the reward function  $h(x)$  (dashed gray) and the value function  $v_f(x, x)$  (solid black). The “pasting point” is  $b_f^* = 2.8845$  (black dot). (c) Plots of the reward function  $H_{f,q}^{(1)}(z)$  (dashed gray) and its smallest concave majorant (solid black), along with the “pasting point”  $\bar{z}_f^* = 0.5441$  (black dot). (d) Plots of the reward function  $v_f(x, x) - h(x) - c$  (dashed gray) and the value function  $v_f^{(1)}(x)$  (solid black), and the “pasting point”  $\underline{b}_f^* = 1.9488$  (black dot).

## 5.1 Value function and optimal strategy

For simplicity, we use identical discounting rate  $q > 0$  for both liquidation and acquisition. We also assume that the transition cost for purchasing one unit of  $X$  is also  $c_0$ . In other words, we set  $c = 2c_0$  in (7). Upon purchasing of the asset, we set a percentage drawdown trailing stop, i.e.  $f(x) = (1 - \alpha)x$ , where  $\alpha \in (0, 1)$  is a constant.

In this study, we select the following parameter values:

$$\lambda = 0.6, \theta = 1, \sigma = 0.2, q = 0.05, c_0 = 0.02, \alpha = 0.3. \quad (44)$$

This means that we will liquidate the asset whenever its price drops from its running maximum since the acquisition by more than 30%.

In Figure 2(a), we plot the function  $H(\cdot)$  defined as in (11). We also have plotted the function  $H_f(\cdot)$  defined as in (32) (see also (39)), which is obtained by first solving equation (30) with  $f(x) = (1 - \alpha)x$  for  $b_f^* (= 2.8845)$ , and then using ODE (39) to numerically obtain  $H_f(\cdot)$ . We notice that, in contrast to the value function for a fixed stop-loss level (Theorem 3.1, see also Leung and Li (2015)), the function  $H_f(\cdot)$  is not concave over  $(0, \psi_q(b_f^*))$ . This is because, although  $\phi_q^-(x)H_f(\psi_q(x)) = v_f(x, x)$  is the value function for the optimal stopping problem (4) when  $x = \bar{x}$ , it does not yield a martingale of  $(X_t, \bar{X}_t)$ , which requires using the function  $v_f(x, \bar{x})$ , not  $v_f(x, x)$ .

In Figure 2(b), we plot the reward function  $h(x)$  and the value function  $v_f(x, x)$  for the optimal liquidation problem (4) with  $x = \bar{x}$ .

In Figure 2(c), we plot the function  $H_{f,q}^{(1)}(z)$  defined in (36) under the current exponential OU model. By checking the function's derivative numerically, we conclude that it is concave to the left of its maximum point. Hence, the smallest concave majorant is given by

$$\hat{H}_{f,q}^{(1)}(z) = \begin{cases} H_{f,q}^{(1)}(z), & \forall z \in (0, \bar{z}_f^*), \\ H_{f,q}^{(1)}(\bar{z}_f^*), & \forall z \in [\bar{z}_f^*, \infty). \end{cases}$$

Therefore, in this case, the optimal acquisition strategy is to purchase the asset once the price is lower than  $\bar{b}_f^* = 1.9488$ .

In Figure 2(d), we plot the function  $v_f(x, x) - h(x) - c$  and the value function  $v_f^{(1)}(x)$  for the optimal acquisition problem (7), and the “pasting point” is at  $\psi_q^{-1}(\bar{z}_f^*) = 1.9488$ .

In summary, for the exponential OU model (43) with parameters as given in (44), the optimal trading strategy is to purchase the asset when price is lower than  $\psi_q^{-1}(\bar{z}_f^*) = 1.9488$ , and setup the 30% trailing stop order as an exit plan, and then wait until either the trailing stop is being activated or the price reaches target  $b_f^* = 2.8845$ .

Lastly, in Figure 3 we plot the early liquidation premium of  $\rho_f \wedge \tau_X^+(b_f^*)$  over the plain trailing stop  $\rho_f$  when  $x = \bar{x}$ . This measure the “value” of our result in problem (4). By Corollary 4.3, we know that

$$p_f(x, x) = \exp\left(-\int_x^{b_f^* \vee x} \frac{\psi_q'(u) du}{\psi_q(u) - \psi_q(f(u))}\right) (h(b_f^* \vee x) - g_f(b_f^* \vee x, b_f^* \vee x)). \quad \forall x \in I. \quad (45)$$

To numerically evaluate (45), we use the fusion of a “limiting order”  $\tau_X^+(b)$  and the trailing stop  $\rho_f$ , with  $b$

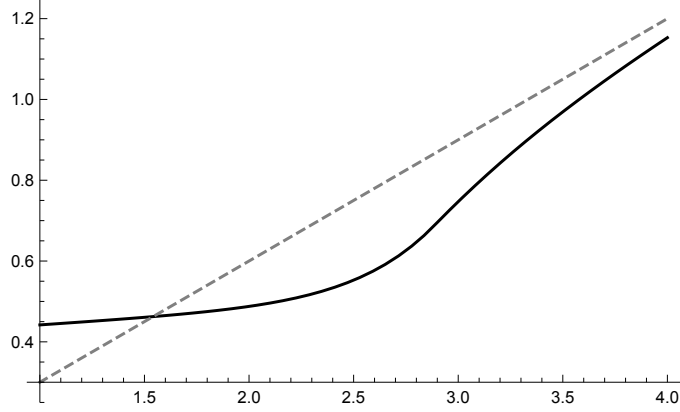


Figure 3: Earlier liquidation premium (black)  $p_f(x, x)$  and function  $x - f(x) = \alpha x$  (dashed) under the exponential OU model (43).

chosen sufficiently large so that

$$\mathbb{E}_x(e^{-q(\tau_X^+(b) \wedge \rho_f)} \mathbf{1}_{\{\tau_X^+(b) < \rho_f\}}) < 0.005, \quad 0 < h(b) \mathbb{E}_x(e^{-q(\tau_X^+(b) \wedge \rho_f)} \mathbf{1}_{\{\tau_X^+(b) < \rho_f\}}) < 0.03,$$

for all  $x$  in the plotting region of Figure 3. Then  $g_f(x, x)$  is approximated by the value of this strategy, which is subsequently solved using an ODE similar as (40).

In Figure 3, we compare the early liquidation premium  $p_f(x, x)$  with the function  $x - f(x) = \alpha x$  ( $\alpha = 0.3$ ), which is the maximum loss of the trailing stop order if the price  $X$  reaches the trailing floor immediately (but without an overshoot). We notice that, for large  $x$ , the gain from our strategy over the plain trailing stop approaches 30% of the price level. Take into account of discounting and transaction costs, this example suggests that setting a trailing stop when the asset price is high will almost always incur a 30% loss at exit.

## 5.2 Sensitivity Analysis

The following illustrative numerical examples will shed light on the sensitivity of the optimal acquisition and liquidation thresholds,  $\underline{b}_f^*$  and  $b_f^*$ , with respect to the trailing stop level  $\alpha$ , and transaction cost  $c_0$ . This involve numerical computation of the thresholds, as well as the critical level where function  $(\mathcal{L} - q)h(x)$  vanishes.

In Figure 4(a), we plot  $(b_f^*, x_0, \underline{b}_f^*)$  as a function of the trailing stop level  $\alpha$ , with  $x_0$  (the dashed line) defined in Lemma 2.2. The optimal liquidation level  $b_f^*$  is increasing in  $\alpha$ , confirming our result in Theorem 4.2. Moreover, the optimal acquisition level  $\underline{b}_f^*$  is also increasing in  $\alpha$ . Recalling that a higher  $\alpha$  means a lower trailing stop trigger, this means that a larger downside protection induces the investor to enter the market earlier. As seen in Figure 4(a), the investor with a higher  $\alpha$  will acquire the asset at a price level closer to the critical level  $x_0$ . Our numerical results also suggest that, for small  $\alpha$ , it may not be optimal to initiate the position at all, because the gain to be realized at the sell order at  $b_f^*$  or at the trailing stop will be too low compared to the transaction cost  $c_0$ . In such cases, we observe that  $\sup_{x \in \mathbb{R}} (v_f(x, x) - h(x)) < c_0 = 0.02$ .

In Figure 4(b), we plot  $(b_f^*, x_0, \underline{b}_f^*)$  as a function of the asset's volatility parameter  $\sigma$ . We see that, as  $\sigma$  increases, the optimal liquidation level increases, thanks to stronger force from the Brownian motion. However, the acquisition price level is lower for higher  $\sigma$ , which means that the investor is willing to establish a position at a lower price. However, higher volatility will increase the likelihood for the asset price to reach

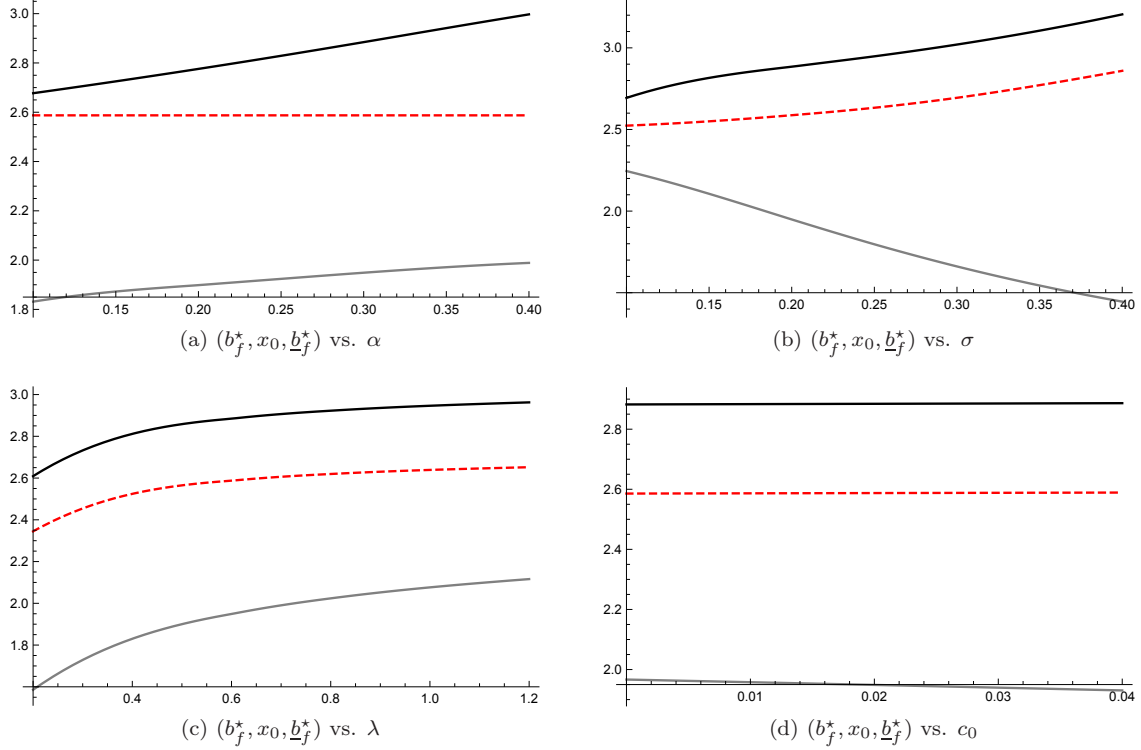


Figure 4: Sensitivity of thresholds  $b_f^*$  (black),  $\underline{b}_f^*$  (gray), and the root  $x_0$  (red dashed) of  $(\mathcal{L} - q)h(x) = 0$ , under the exponential OU model (43): (a) Dependence on  $\alpha \in [0.1, 0.4]$ ; (b) Dependence of on  $\sigma \in [0.1, 0.4]$ ; (c) Dependence of on  $\lambda \in [0.2, 1.2]$ , and (d) Dependence of on  $c_0 \in [0, 0.04]$ . In all figures, other parameters are set as in (44).

low levels earlier, so the actual entry time by the investor may be earlier or later. The decreasing pattern of  $\underline{b}_f^*$  with respect to  $\sigma$  suggests that the investor voluntarily lowers the take-profit level to mitigate the risk of realizing a reduced profit or a loss at the trailing stop in a more volatile market.

Figure 4(c) illustrates the effect of the asset's rate of mean reversion  $\lambda$ . A higher  $\lambda$  means that the log-price will move around its long-term mean  $\theta$  faster. As a response, the investor enters the market earlier at a higher entry level and exit at a lower level, resulting in a quick roundtrip, as reflected in the plot by the increasing trends of  $b_f^*$  and  $\underline{b}_f^*$  with respect to  $\lambda$ . Moreover, their distance is shrinking as  $\lambda$  continues to increase. Intuitively, since the asset price tends to rapidly revert to the mean, it does not make sense to select entry and exit price levels that are far apart and away from the mean as the chance of execution is too low.

The effect of transaction cost  $c_0$  is shown in Figure 4(d), where we plot  $(b_f^*, x_0, \underline{b}_f^*)$  as a function of  $c_0$ . The optimal liquidation (resp. acquisition) level  $b_f^*$  ( $\underline{b}_f^*$ , resp.) increases (decreases, resp.) slightly with respect to  $c_0$ . To interpret, higher transaction costs discourage both acquisition and liquidation, though the effect is not significant. Nevertheless, as pointed out in our analysis above, while there is always a finite optimal liquidation price  $b_f^*$  given any transaction cost, a high transaction cost may make the trade unprofitable and thus exclude market entry.



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